定理 $\det(A) = \det(A^T)$

证: $\det(A^T) = \sum_{(i_1\cdots i_n)\in S_n} (-1)^{\tau(i_1\cdots i_n)} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}$

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ 1 & 2 & \cdots & n \end{pmatrix} \leftrightarrow \cdots \leftrightarrow \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

$$\Rightarrow \begin{cases} a_{i_1 1} \cdots a_{i_n n} = a_{1 j_1} a_{2 j_2} \cdots a_{n j_n} \\ (-1)^{\tau(i_1 \cdots i_n)} = (-1)^{\tau(j_1 \cdots j_n)} \end{cases}$$

$$\Rightarrow \det(A^T) = \sum_{(j_1\cdots j_n)\in S_n} (-1)^{\tau(j_1\cdots j_n)} a_{1 j_1} \cdots a_{n j_n} = \det(A). \quad \square$$

定理(列展开) $A = (a_{ij})_{n\times n}$.

$$\det(A) = \sum_{i=1}^{n} a_{ik} A_{ik} = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} M_{ik}$$

定理: $\det(AB) = \det A \cdot \det B$.

证: $\det(AB) = \det\left(\sum_{j_1=1}^{n} a_{1 j_1} \beta_{j_1}, \cdots, \sum_{j_n=1}^{n} a_{n j_n} \beta_{j_n} \right)$ $\square$

例 $A \in F^{m\times n}$, $B \in F^{n\times m}$, $m > n \Rightarrow \det(AB) = 0$.

证: $AB = (A, 0)\cdot\begin{pmatrix} B \\ 0 \end{pmatrix}$ $\det(AB) = \det(A, 0)\cdot\det\begin{pmatrix} B \\ 0 \end{pmatrix} = 0$

$\square$

例： $A\in F^{n\times n}$, $B\in F^{n\times m}$, $C\in F^{m\times n}$, $D\in F^{n\times n}$, $\underbrace{A \text{可逆}}_{D\text{可逆?}}$ 则

$$\det\begin{pmatrix} A & C \\ B & D \end{pmatrix} = \det(A)\cdot\det(D-BA^{-1}C)$$

1° $\left.\begin{matrix} m=n \\ AB=BA \end{matrix}\right\} \Rightarrow A(D-BA^{-1}C) = AD-ABA^{-1}C = AD-BAA^{-1}C = AD-BC$

$$\Rightarrow \det\begin{pmatrix} A & C \\ B & D \end{pmatrix} = \det(AD-BC)$$

2° $\left.\begin{matrix} m=n \\ AC=CA \end{matrix}\right\} \Rightarrow A(B-BA^{-1}C) = A(D-BA^{-1}C) = AB - ABA^{-1}CAA^{-1}$

$$= AD - ABA^{-1}ACA^{-1}$$

$$= AD - ABCA^{-1}$$

$$= A\cdot(DA-BC)A^{-1}$$

$$\Rightarrow \det\begin{pmatrix} A & C \\ B & D \end{pmatrix} = \det(DA-BC)$$

定理： $A=(a_{ij})_{n\times n}$ $A_{ij} := a_{ij}$ 的代数余子式

$$A^* := \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} = (A_{ij})_{n\times n}^T$$

A 的伴随矩阵

则

$$A^*A = AA^* = \det(A)\cdot I_{(n)}$$

即

$$\sum_{k=1}^{n} a_{ik}A_{jk} = \det(A)\delta_{ij}$$

&

$$\sum_{k=1}^{n} a_{ki}A_{kj} = \det(A)\cdot\delta_{ij}$$

其中 $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$ Kronecker 记号 i.e. $I_{(n)} = (\delta_{ij})_{n\times n}$

证：

$$\text{展开}\rightarrow \begin{array}{c} \begin{pmatrix} a_{i1} & \cdots & a_{in} \\ a_{i1} & \cdots & a_{in} \end{pmatrix} \end{array} \begin{array}{c} i \\ j \end{array}$$

□

定理：A 为 n 阶方阵. 则

$$\underset{①}{A \text{ 可逆}} \Leftrightarrow \underset{②}{\det(A)\neq 0} \Rightarrow \underset{③}{A^{-1} = \frac{1}{\det(A)} A^*}.$$

证：① ⇒ ②：$AA^{-1} = I_{(n)} \Rightarrow \det(A)\det(A^{-1}) = 1$

$\Rightarrow \det(A) \neq 0.$

② ⇒ ① ③：$A\cdot\left(\frac{1}{\det(A)}A^*\right) = I_{(n)} = \left(\frac{1}{\det(A)}A^*\right)\cdot A$

$\Rightarrow$ A 可逆 且 $A^{-1} = \frac{1}{\det(A)} A^*$ □

A 可逆 $\overset{def}{\Longleftrightarrow}$ $\exists X$ s.t. $AX = I = XA$ ①

$\Leftrightarrow \exists X$ s.t. $AX = I$ ②

$\Leftrightarrow \exists X$ s.t. $XA = I$ ③

① ⇒ ②, ③ ✓ ② $\overset{?}{\Rightarrow}$ ① $AX = I \Rightarrow \det A \neq 0$

$\Rightarrow$ A 可逆 ③

§ 行列式计算

例：(1)

$$\begin{vmatrix} 0 & 5 & -4 & 5 \\ -3 & -1 & -5 & 3 \\ 3 & 1 & -2 & -3 \\ -1 & 4 & -5 & -1 \end{vmatrix} \xrightarrow[3r_4 \to r_3]{r_3 \to r_2} \begin{vmatrix} 0 & 5 & -4 & 5 \\ 0 & 0 & -7 & 0 \\ 0 & 13 & -17 & -6 \\ -1 & 4 & -5 & -1 \end{vmatrix} \xrightarrow{-c_4 \to c_3} \begin{vmatrix} 0 & 0 & -4 & 5 \\ 0 & 0 & -7 & 0 \\ 0 & 19 & -17 & -6 \\ -1 & 5 & -5 & -1 \end{vmatrix}$$

$$(-1)^{1+4} \cdot (-1)^{1+3} \cdot (-1) \cdot 19 \cdot \begin{vmatrix} -4 & 5 \\ -7 & 0 \end{vmatrix} = 19 * 35 = 665$$

(2)

$$\begin{vmatrix} x & 1 & \cdots & 1 \\ 1 & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & x \end{vmatrix} = \begin{vmatrix} x+n-1 & 1 & \cdots & 1 \\ x+n-1 & x & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ x+n-1 & 1 & \cdots & x \end{vmatrix} = (x+n-1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & x & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & & x \end{vmatrix}$$

$$= (x+n-1) \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & x-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 1 & 0 & \cdots & x-1 \end{vmatrix} = (x+n-1) \cdot (x-1)^{n-1}$$

(3)

$$\begin{vmatrix} x & & & & & & a_0 \\ -1 & \ddots & & & & & \vdots \\ & \ddots & x & & & & \vdots \\ & & -1 & x & & & a_{i-1} \\ & & & -1 & x & & \vdots \\ & & & & \ddots & \ddots & \vdots \\ & & & & & -1 & x+a_{n-1} \end{vmatrix} = \sum_{i=1}^{n-1} (-1)^{n+i} a_{i-1} \underbrace{\overbrace{\begin{vmatrix} x & & \\ -1 & \ddots & \\ & -1 & x \end{vmatrix}}^{(i-1)\times(i-1)} \begin{vmatrix} -1 & x & \\ & \ddots & x \\ & & -1 \end{vmatrix}}_{(n-i)\times(n-i)} + (x+a_{n-1}) \cdot x^{n-1}$$

$$= x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

④

(4) $$\overset{\curvearrowright_{-a_n}\ \curvearrowright_{-a_n}\ \cdots}{\begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{vmatrix}} = \begin{vmatrix} 1 & a_1-a_n & a_1^2-a_1a_n & \cdots & a_1^n-a_1^{n-1}a_n \\ 1 & a_2-a_n & a_2^2-a_2a_n & \cdots & a_2^n-a_2^{n-1}a_n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underset{0}{\underset{\shortparallel}{a_n-a_n}} & \underset{0}{\underset{\shortparallel}{a_n^2-a_na_n}} & \cdots & \underset{0}{\underset{\shortparallel}{a_n^n-a_n^{n-1}a_n}} \end{vmatrix}$$

$\shortparallel$

$\Delta_n(a_1,\cdots,a_n)$

$$= (-1)^{n+1}(a_1-a_n)\cdots(a_{n-1}-a_n)\begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n-1} & \cdots & a_{n-1}^{n-1} \end{vmatrix} =: \Delta_{n-1}(a_1,\cdots,a_{n-1})$$

$$\Rightarrow \Delta_n(a_1,\cdots,a_n) = \prod_{i=1}^{n-1}(a_n-a_i)\cdot\Delta(a_1,\cdots,a_{n-1}) = \prod_{1\le i<j\le n}(a_j-a_i)$$

例: $$\Delta_n = \begin{vmatrix} 2 & 1 & & & \\ 1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & 1 \\ & & & 1 & 2 \end{vmatrix}_{n\times n}$$ $\Delta_1 = 2$, $\Delta_2 = 2\times 2-1 = 3$

$$\Delta_n = 2\Delta_{n-1} - \Delta_{n-2} \Rightarrow \Delta_n - \Delta_{n-1} = \Delta_{n-1} - \Delta_{n-2} = \cdots = \Delta_2 - \Delta_1 = 1$$
$$\Rightarrow \Delta_n = (n-1) + \Delta_1 = n+1$$